# RadonAna: A C-based Package for Analytically Based Radon Transformation

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**Abstract**

In this report the Radon transform of some geometrical primitives are found. Using the linearity of the Radon transform and basic rules about translation, rotation and scaling rather complicated two dimensional functions can be modeled and the corresponding Radon transform can expressed on an analytical form. This provides a useful tool to generate test sets for testing of reconstruction algorithms.

A program is available from the author, where the presented theory is implemented.

# Contents

# 1 The Radon Transform

The linear Radon transform is a mathematical transformation of a two dimensional signal into a new two dimensional signal. The transform is essential for reconstruction of tomographical images. The choice of parameters used in the following matches the definition used in CT[1] and PET[2].

The Radon transform of a signal $g(x, y)$ is given by

$$\breve{g}(\rho, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)\ \delta(\rho - x\cos\theta - y\sin\theta)\ dx\ dy \tag{1}$$

where $\delta(\cdot)$ is the Dirac delta function. The Radon transform can also be expressed as a line integral

$$\breve{g}(\rho, \theta) \quad = \quad \int_{-\infty}^{\infty} g(\rho\cos\theta - s\sin\theta, \rho\sin\theta + s\cos\theta)\ ds \tag{2}$$

The key idea shown in Equation 1 and 2 is to integrate $g(x, y)$ along the line with parameters $\rho$ and $\theta$.

In tomographical applications the inverse Radon transform is used to find the linear absorption coefficient (CT-scanners) within the, e.g., brain. In PET scanners the activity of the brain can reconstructed using the inverse Radon transform. When implementing inverse Radon transform algorithms it is very relevant to have analytical Radon transform pairs which enables quantitatively analysis of the the numerical behavior of the algorithms. In this report a set of simple geometrical signals *primitives*, which can be Radon transformed, is presented. These can be combined to generate more complex signals, which still can be Radon transformed analytically. A program is available to generate sampled versions of the image and corresponding Radon transform based on analytically Radon transformation of the primitives.

## 1.1 The Circular Disc

One of the basic shapes is the circular disc, when working with the linear Radon transform. The disc with uniform excitation 1 and radius is set to 1 is shown on Figure 1.

This rotational symmetric function can be written as

$$g_c(x, y) = \mu(1 - x^2 - y^2) \tag{3}$$

where $\mu\cdot$ is the Hamilton spring function. The rotational symmetry implies that the Radon transform of this function can be found at any angle, e.g. $\theta = 0$ as shown on Figure 2.

As indicated on Figure 2 is the Radon transform merely the length across the disc. Assume that the line lies in the distance $\rho$ from the $(0, 0)$. With simple geometry it is found that

$$\breve{g}_c(\rho, \theta) = \begin{cases} 2\sqrt{1 - \rho^2} & \text{if } \rho^2 < 1 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

This result has been used extensively together with the properties shown in A. This implies that the Radon transform of a sum of ellipses can be found analytically. For many years the standard test image has been the Shepp Logan head Phantom shown on Figure 3 and the corresponding Radon transform is shown on Figure 4.

[1] Computerized Tomography

[2] Positron Emission Tomography

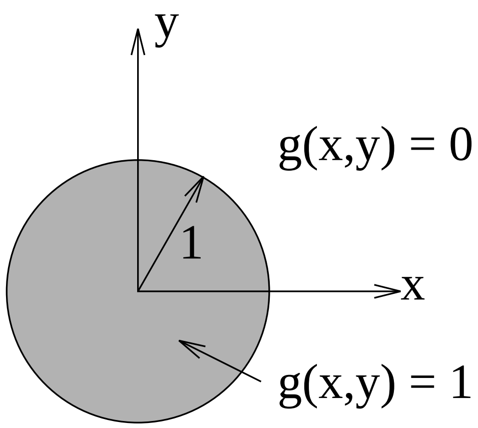


Figure 1: The circular disc.

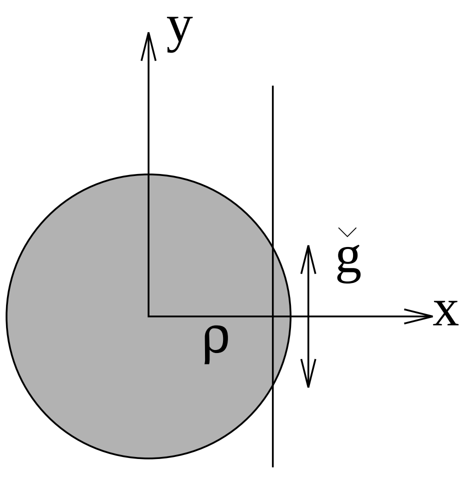


Figure 2: The Radon transform of the circular disc is the linear length across the disc.

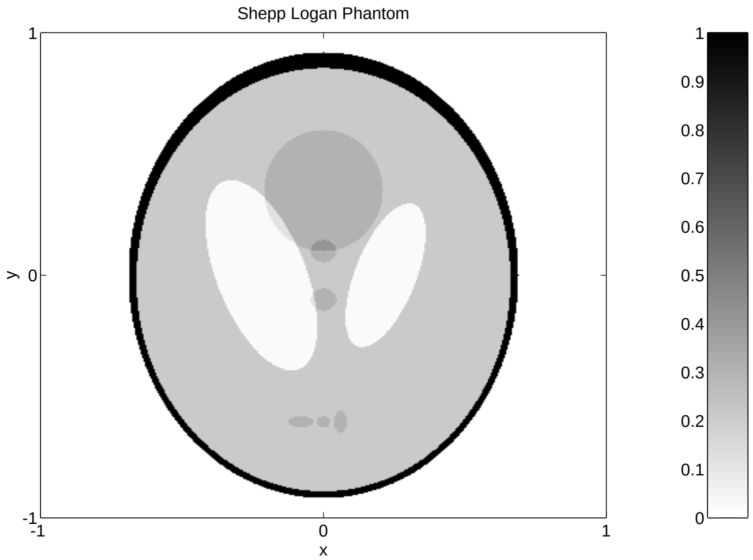


Figure 3: The Phantom brain

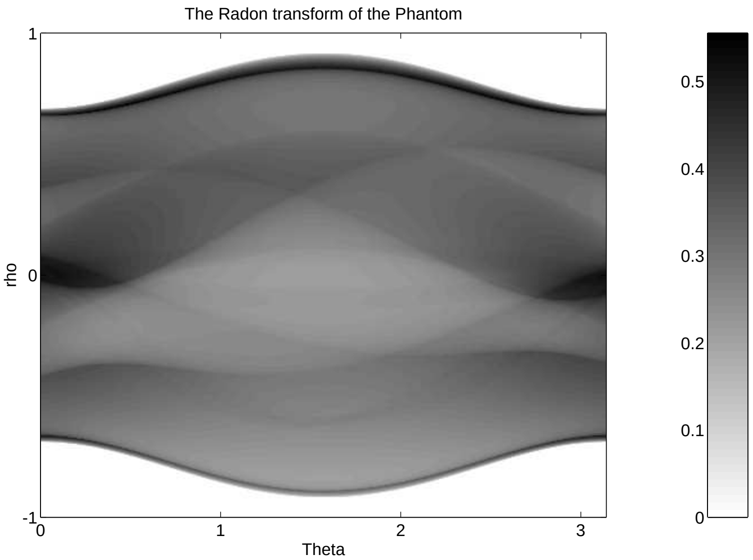


Figure 4: The corresponding Radon transform

## 1.2 The Square

Another very useful primitive is the square as shown on Figure 5.

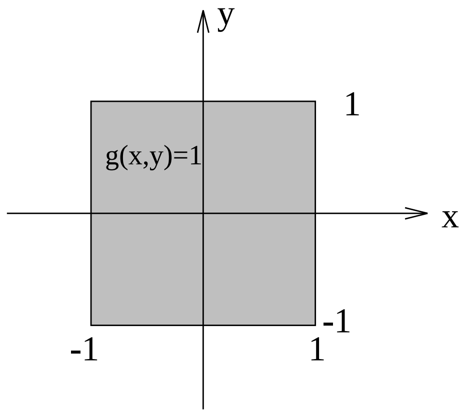


Figure 5: The square.

This function can be written as

$$g_s(x,y) = \mu(1-|x|)\mu(1-|y|) \tag{5}$$

It is easy to see that then Radon transform of this primitive must exhibit several symmetries.

$$\begin{aligned} g_s(x,y) &= g_s(|x|,|y|) && (6)\\ \check{g}_s(\rho,\theta) &= \check{g}_s(\rho,\theta+p\frac{\pi}{4}) && (7)\\ &= \check{g}_s(\rho,\frac{\pi}{4}-\theta) && (8) \end{aligned}$$

where $p$ is an integer. Equation 7 implies that the Radon transform is only needed in the interval $[0;\pi/2]$. Equation 8 further sharpens this demand to the interval $[0;\pi/4]$. Using the general feature $\check{g}(\rho,\theta) = \check{g}(-\rho,\theta+\pi)$ implies together with Equation 7 that only $\rho \geq 0$ must be investigated. It is also easy to find that

$$\check{g}_s(\rho,\theta) = 0 \text{ if } \rho > \sqrt{2} \tag{9}$$

The remaining case $\rho < \sqrt{2}$ is fairly easy due to the simple geometry. First are the intersections between the line $\rho = x\cos\theta + y\sin\theta$ and the boundaries found

$$\begin{aligned} y=-1 &\Rightarrow x = x_{-1} = \frac{\rho}{\cos\theta} + \tan\theta && (10)\\ y=1 &\Rightarrow x = x_1 = \frac{\rho}{\cos\theta} - \tan\theta && (11)\\ x=1 &\Rightarrow y = y_1 = \frac{\rho}{\cos\theta} - \cot\theta && (12) \end{aligned}$$

Three cases exist

$$x_1 > 0 \Rightarrow \check{g}(\rho,\theta) = 0 \tag{13}$$

$$x_1 < 1 \text{ and } x_{-1} < 1 \quad \Rightarrow \tag{14}$$
$$\breve{g}(\rho,\theta) = \sqrt{2^2 + (x_1 - x_{-1})^2} \tag{15}$$
$$= \frac{2}{\cos\theta} \tag{16}$$

$$x_1 < 1 \text{ and } x_{-1} > 1 \quad \Rightarrow \tag{17}$$
$$\breve{g}(\rho,\theta) = \sqrt{(1-x_1)^2 + (1-x_{-1})^2} \tag{18}$$

These formulas are computed in their respective region giving the total Radon transform of the square. On Figure 6 is the square shown and Figure 7 shown the corresponding Radon transform.

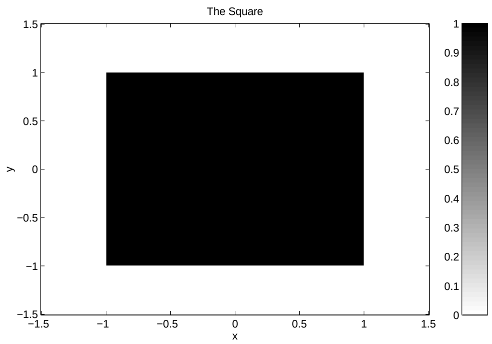


Figure 6: The "Square".

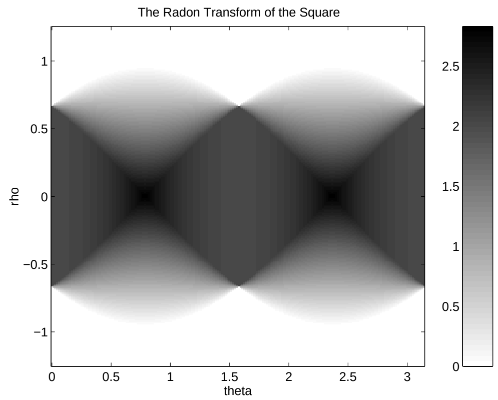


Figure 7: The Radon transform of the "Square".

## 1.3 The Triangle

The following primitive is the triangle shown on Figure 8.

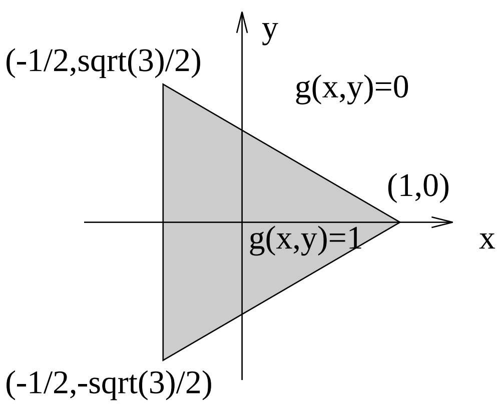


Figure 8: The triangle.

This Radon transform of this primitive can also calculated on a analytical form. The first thing is that the rotational symmetry implies that

$$\breve{g}(\rho,\theta) = \breve{g}(\rho,-\theta) = \breve{g}\left(\rho,\theta+p\frac{2\pi}{3}\right) \tag{19}$$

where $p$ is an integer. These relation together with $\breve{g}(\rho,\theta) = \breve{g}(-\rho,\theta+\pi)$ implies that it is only relevant to investigate $\rho \geq 0$ and $0 \geq \theta < \frac{\pi}{3}$.

Three lines are relevant

- $l_-$ : $y = -\frac{1}{\sqrt{3}}(x-1)$, i.e., the upper slanted line of the triangle.
- $l_+$ : $y = \frac{1}{\sqrt{3}}(x-1)$, i.e., the lower slanted line of the triangle.
- $l$ : $\rho = x\sin\theta + y\sin\theta$, i.e., the transformation line.

First is the intersection between $l$ and $l_+$ is found, i.e.,

$$x_+ = \frac{\sqrt{3}\rho + \sin\theta}{\sqrt{3}\cos\theta + \sin\theta} \tag{20}$$

$$y_+ = \frac{1}{\sqrt{3}}(x_+ - 1) \tag{21}$$

and the intersection between $l$ and $l_-$

$$x_- = \frac{\sin\theta - \sqrt{3}\rho}{\sin\theta - \sqrt{3}\cos\theta} \tag{22}$$

$$y_- = -\frac{1}{\sqrt{3}}(x_- - 1) \tag{23}$$

Another interesting intersection is between the vertical line $x = -\frac{1}{2}$ and the integration line. The corresponding $y = y_l$ is given by

$$y_l = \frac{2\rho + \cos\theta}{2\sin\theta} \tag{24}$$

It is easy to realize that $x_+ > 0$ implies that the integration line does not cross the triangle, i.e., $\breve{g}(\rho,\theta) = 0$.

The remaining cases all imply that the integration line intersects $l_+$

$$y_l < \frac{\sqrt{3}}{2} \quad \Leftrightarrow \quad \breve{g}(\rho,\theta) = \sqrt{\left(x_+ + \frac{1}{2}\right)^2 + (y_l - y_+)^2} \tag{25}$$

$$y_l > \frac{\sqrt{3}}{2} \quad \Leftrightarrow \quad \breve{g}(\rho,\theta) = \sqrt{(x_+ + x_-)^2 + (y_+ - y_-)^2} \tag{26}$$

These formulas are computed in their respective region giving the total Radon transform of the triangle. On Figure 9 is the square shown and Figure 10 shown the corresponding Radon transform.

Using the presented set of primitives with proper scaling, rotation and translation as shown in Appendix A, e.g., a "man" can be created as shown on Figure 11. The corresponding Radon transform is shown on Figure 12.

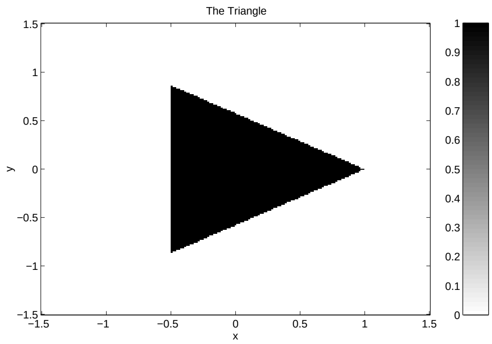


Figure 9: The “Triangle”.

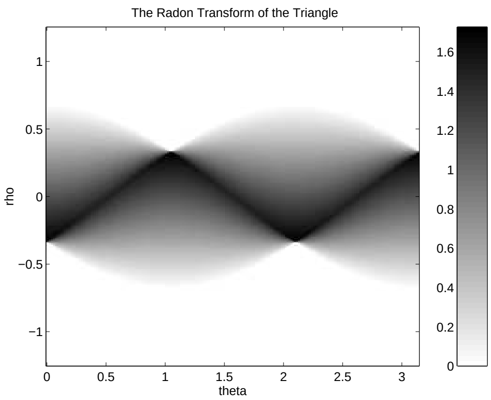


Figure 10: The Radon transform of the “Triangle”.

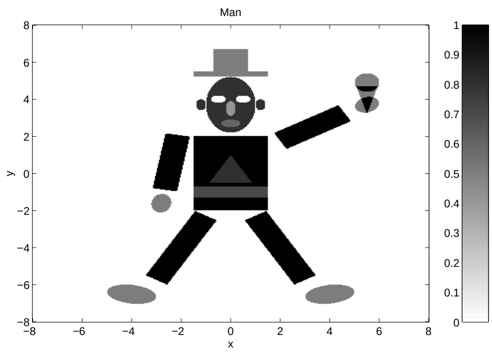


Figure 11: The “Man”.

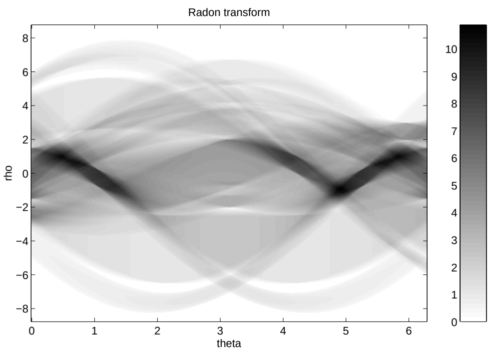


Figure 12: The Radon transform of the “Man”.

## 1.4 The Gaussian Bell

The previous cases all concern finite objects with sharp edges. The next case is the infinite and soft varying Gaussian bell

$$g(x,y) = \exp(-x^2 - y^2) \tag{27}$$

Due to the rotational symmetry the Radon transform can be calculated at any angle, e.g., $\theta = 0$, where the integration reduces to

$$\begin{aligned} \check{g}(\rho,\theta) &= \int_{-\infty}^{\infty} g(x=\rho,y)\ dy & (28)\\ &= \exp(-\rho^2) \int_{-\infty}^{\infty} \exp(-y^2)\ dy & (29)\\ &= \sqrt{\pi}\exp(-\rho^2) & (30) \end{aligned}$$

The Gaussian bell is shown on Figure 13 and the corresponding Radon transform given in Equation 30 is shown on Figure 14. Note that the bell is infinite, thus it cannot be bounded in the image- or the Radon-domain.

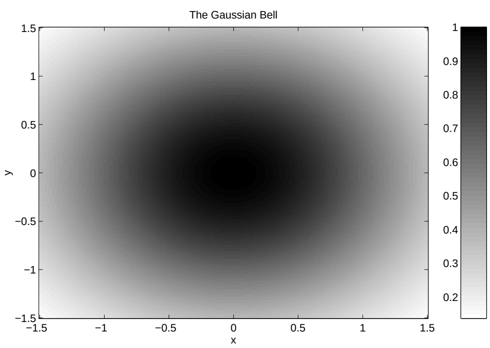


Figure 13: The Gausian Bell.

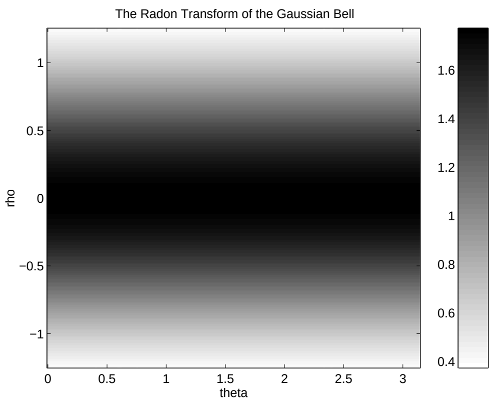


Figure 14: The Radon transform of the Gaussian Bell.

## 1.5 The "Pyramid"

The square can be used to simulate digital images using pixels with constant excitation. This is in fact a nearest neighbor approach. In many application, e.g., Radon transformation, the nearest neighbor approach introduces to much noise. In this case a linear interpolation strategy is very common. The discrete image is convoluted with a triangle in both directions. If normalizing the sampling distance to 1, the convolution kernel can be written as

$$g(x,y) = \begin{cases} (1-|x|)(1-|y|) & \text{for } |x|<1, |y|<1 \\ 0 & \text{otherwise} \end{cases} \tag{31}$$

and the interpolated image can written as the integral

$$s_i(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_m \sum_n s(m,n)\ \delta(x'-x_m)\ \delta(y'-y_n)\ g(x-x',y-y')\ dx'\ dy' \tag{32}$$

$$= \sum_m \sum_n s(m,n)\ g(x - x_m, y - y_n) \tag{33}$$

where $s(m,n)$ is the discrete image. Due to the linearity and translation properties shown in Appendix A the Radon transform of the continuous function $s_i(x,y)$ can found if the Radon transform of the kernel $g(x,y)$ can be found. This will be done.

Due to the symmetry in the pyramid-like kernel it is easy to find that

$$\check{g}(\rho,\theta) = \check{g}(-\rho,\theta) = \check{g}(\rho,\theta + p\pi/2) = \check{g}(\rho, pi/2 - \theta) \tag{34}$$

It is also apparent that the Radon transform is zero if $\rho > \sqrt{2}$, i.e., the Radon transform is only needed in the interval $0 < \rho < \sqrt{2}$ and $0 \leq \theta < \pi/4$.

$$\check{g}(\rho,\theta) = \frac{1}{|\cos\theta|} \int_{-\infty}^{\infty} g\left(\frac{\rho}{\cos\theta} - y\tan\theta, y\right) \tag{35}$$

$$= \frac{1}{|\cos\theta|} \int_{y^*}^{1} (\cos\theta - |\rho - y\sin\theta|)(1 - |y|)\ dy \tag{36}$$

where

$$y^* = \max\{-1, y_1\} \tag{37}$$

and

$$y_1 = \frac{\rho - \cos\theta}{\sin\theta} \tag{38}$$

For test purposes it can used that $y_1 > 1$ implies that $\check{g}(\rho,\theta) = 0$. The meaning of $y_1$ is indicated on Figure 15.

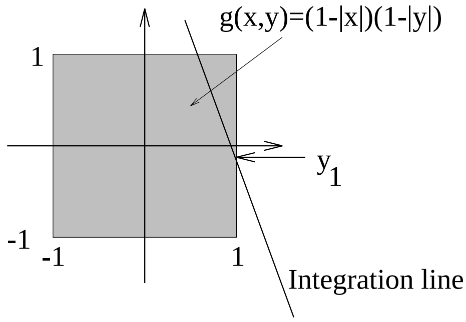


Figure 15: The "Pyramid".

Yet another definition. It turns out that a vital parameter is

$$y_0 = \frac{\rho}{\sin\theta} \tag{39}$$

Solving the rather simple integrals in Equation 36 (it only requires good bookkeeping) the Radon transform can be derived. The result can be written as

$y_1 < -1$ and $y_0 > 1$

$$\check{g}(\rho,\theta) = \frac{3\cos\theta - \rho}{\cos^2\theta} \tag{40}$$

$y_1 < -1$ and $y_0 < 1$

$$\breve{g}(\rho,\theta) = \frac{9\cos\theta - \sin\theta - 3\rho y_0 + \rho y_0^2}{3\cos^2\theta} \tag{41}$$

$-1 < y_1 < 0$ and $y_0 > 1$

$$\breve{g}(\rho,\theta) = \frac{\cos\theta(9 - 3y_1 + 5y_1^2) - 3\rho + 3\rho y_1 + \rho y_1^2 + \sin\theta}{6\cos^2\theta} \tag{42}$$

$-1 < y_1 < 0$ and $y_0 < 1$

$$\breve{g}(\rho,\theta) = \frac{\cos\theta(9 - 3y_1 + 5y_1^2) + \rho(3 - 3y_0 + 2y_0^2 + 3y_1 + y_1^2) - \sin\theta}{6\cos^2\theta} \tag{43}$$

$y_1 > 0$ and $y_0 > 1$

$$\breve{g}(\rho,\theta) = \frac{\cos\theta(9 - 3y_1 - 5y_1^2) + \rho(3y_1 - 3 - y_0 - y_1^2) + \sin\theta}{6\cos^2\theta} \tag{44}$$

$y_1 > 0$ and $y_0 < 1$

$$\breve{g}(\rho,\theta) = \frac{\cos\theta(9 - 3y_1 - 5y_1^2) + \rho(3 - 6y_0 + 3y_1 + 2y_0^2 - y_1^2) - \sin\theta}{6\cos^2\theta} \tag{45}$$

On Figure 16 is shown the basis "pyramid"-function $g(x, y)$ and on Figure 17 the corresponding Radon transform.

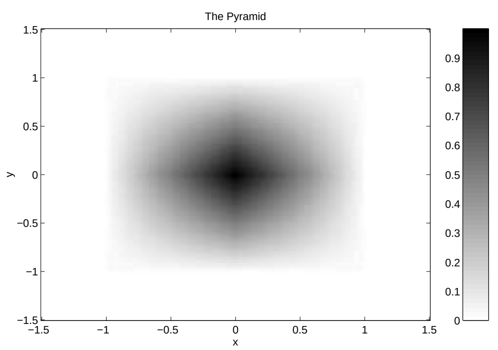


Figure 16: The "Pyramid".

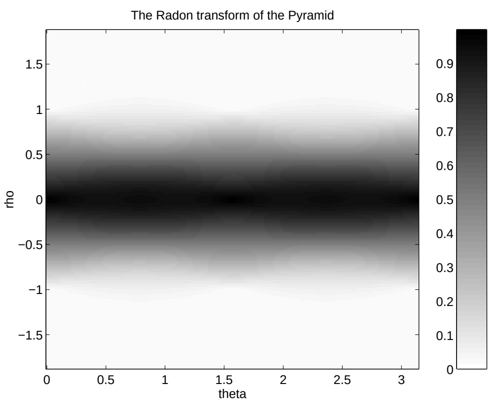


Figure 17: The Radon transform of the "Pyramid".

Note that the Radon transform could also be calculated from the square using that the auto correlation function can be Radon transformed.

# 2 Program Usage

From the author a program `RadonAna` is available which have been used to generate Radon transform pairs. The basic idea is to specify a set of scaled, rotated, and shifted primitives from which the Radon transform is calculated and sampled in both the image domain and the Radon domain (sinogram).

The program uses the following parameters

`OutFileName` [String] With this parameter the base file name is determined. If `OutFileName` is `man.fif` then the image is written in `mana.fif` and the sinogram in the file `manr.fif`.

`XSamples` [Integer] Specifies the number of image samples on the first axis. Should be an odd number.

`YSamples` [Integer] Specifies the number of image samples on the second axis. Should be an odd number. If not given the program uses `XSamples=YSamples`.

`DeltaX` [Float] Specifies the sampling distance of both axes in the image.

`Xmin` [Float] Specifies the minimum sample position in the image on the first axis. If not given the image is centered around the middle.

`Ymin` Specifies the minimum sample position in the image on the second axis. If not given the image is centered around the middle.

`RhoSamples` [Integer] Specifies the number of samples in the distance parameter $\rho$ in the sinogram. Should be an odd number.

`ThetaSamples` [Integer] Specifies the number of samples in the angular parameter $\theta$ in the sinogram. The sinogram is samples linearly from 0 to (approximately) $\pi$ radians. The sampling distance in $\theta$ is $\pi$/`ThetaSamples`.

`DeltaRho` [Float] Specifies the sampling distance in $\rho$. The program will center the sampling points around 0.

`OverSamp` [Integer] This parameter specifies whether an oversampled image and sinogram should be generated or not `OverSamp`= 0. If, e.g., `OverSamp=2` an image is calculated with 5 times (2*`OverSamp`+1) the resolution in both parameters. This image is then averaged so the output value make a better approximation to the average value under the sample. This technique is also applied the the sinogram. It will reduce aliasing problems.

`GenerateImage` [Integer] If set to 1 the image is generated.

`GenerateRadon` [Integer] If set to 1 the sinogram is generated.

`NumberOfShapes` [Integer] Here the number of primitives are specified.

`DebugNiveau` [String] Log files is generated if this parameter is set to `_Debug`. No log-file and no screen output can be used with `DebugNiveau=_HardCore`.

The ordering of these parameters is arbitrary. After these parameters a number of lines must follow specifying a scaling, rotation, and shifting parameters of each primitive. The line uses the following parameters in this order

`Shape`$i$ [String] where $i$ is 0 to `NumberOfShapes`-1.

`Type` [Integer] The type of primitive, where

- **1** A circle centered around (0,0) with radius 1 and uniform signal 1 on the disk.
- **2** A square centered around (0,0). $-1 < x < 1$ and $-1 < y < 1$. The signal is 0 on the disk.
- **3** A Gaussian bell $\exp(-x^2 - y^2)$.
- **4** An even sided triangle centered around $(0, 0)$. The triangle has corners $(1, 0)$, $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$.
- **5** The image and the Radon domain is con-terminated with white Gaussian noise. Note that in this case the noise term in the Radon domain is NOT the Radon transform of the noise in the image.
- **6** A "pyramid" $f(x, y) = (1 - |x|)(1 - |y|)$ for $-1 < x < 1$ and $-1 < y < 1$.

`a` [Float] Scaling of the primitive in the direction of the first axis.

`b` [Float] Scaling of the primitive in the direction of the second axis.

`x0` [Float] Shift of the primitive in direction of the first axis.

`y0` [Float] Shift of the primitive in direction of the second axis.

`phi` [Float] Rotation of the primitive measured in degrees.

`power` [Float] Scaling of the intensity of the primitive.

`Description` [String] Optional description of the primitive.

The program is executed with the initialization file or `.INI`-file as sole argument. A typical `.INI`-file could be the one used to generate the man. This example is shown in Appendix B.

# A Basic Properties of the Radon Transformation

In this appendix several basic properties of the Radon transform are shown.

## A.1 Linearity

The first property is that the Radon transform is a linear transform. If $\alpha_q$ is an array of constants, then

$$g(x,y) = \sum_q \alpha_q \; g_q(x,y) \Rightarrow \breve{g}(\rho,\theta) = \sum_q \alpha_q \; \breve{g}_q(\rho,\theta) \tag{46}$$

The results is that the sinogram of a sum of primitives is the sum the individual sinograms. The linearity has already been exploited to form Figure 11 and Figure 3.

## A.2 Shifting

Assume

$$\begin{aligned} h(x,y) &= g(x-x_0, y-y_0) \Rightarrow & (47)\\ \breve{h}(\rho,\theta) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x-x_0,y-y_0)\;\delta(\rho - x\cos\theta - y\sin\theta)\;dx\;dy & (48)\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\tilde{x},\tilde{y})\;\delta((\rho - x_0\cos\theta - y_0\sin\theta) - \tilde{x}\cos\theta - \tilde{y}\sin\theta)\;d\tilde{x}\;d\tilde{y} & (49)\\ &= \breve{g}(\rho - x_0\cos\theta - y_0\sin\theta, \theta) & (50) \end{aligned}$$

Note that only the $\rho$-coordinate is changed. On Figure 18 the shifted triangle is illustrated and on Figure 19 is the corresponding Radon transform shown.

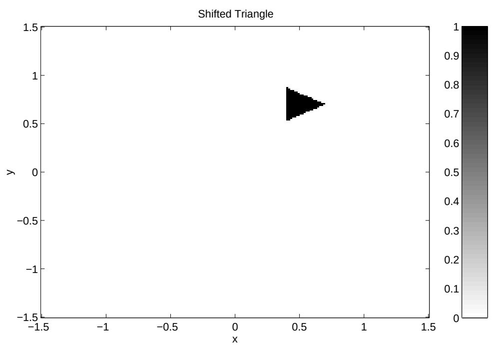


Figure 18: Shifting a triangle.

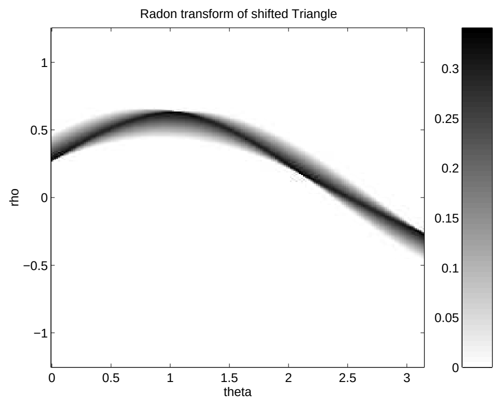


Figure 19: The Radon transform of a shifted triangle.

## A.3 Rotation

Here $g(x,y)$ is expressed in polar form, i.e., $g(x,y) = g(r,\phi)$. In this case rotation is fairly easy

$$h(r,\phi) = g(r, \phi - \phi_0) \tag{51}$$

$$\check{h}(\rho,\theta) = \int_{-\infty}^{\infty}\int_0^{\pi} g(r,\phi-\phi_0)\ \delta(\rho - r\cos\phi\cos\theta - r\sin\phi\sin\theta)\ |r|\ d\phi\ dr \tag{52}$$

$$= \int_{-\infty}^{\infty}\int_0^{\pi} \breve{g}(r,\tilde{\phi})\ \delta(\rho - r\cos(\theta-\phi-\phi_0))\ |r|\ d\phi\ dr \tag{53}$$

$$= \breve{g}(\rho,\theta-\phi_0) \tag{54}$$

This is quite obvious. If the coordinate system $(x, y)$ is turned $\phi_0$, then the Radon transform is also turned $\phi_0$. This is illustrated on Figure 20 and 21, where a small triangle has been rotated and then shifted with the same amount as in Figure 18.

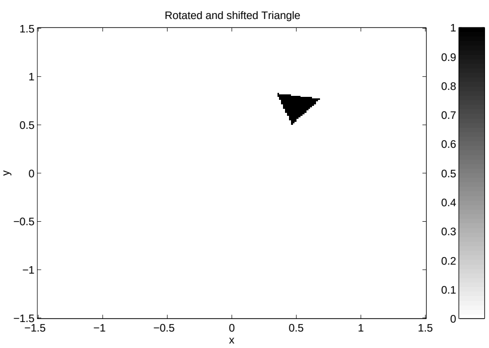


Figure 20: A small triangle rotated and shifted.

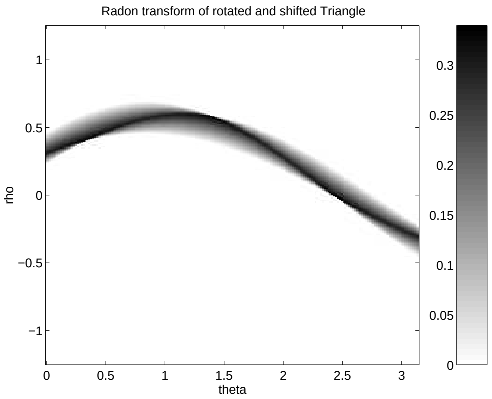


Figure 21: The corresponding Radon transform

## A.4 Scaling

Assume a scaling in both coordinates

$$h(x,y) = g\left(\frac{x}{a},\frac{y}{b}\right),\ \ a > 0 \text{ and } b > 0 \tag{55}$$

In this case the Radon transform can be rewritten

$$\check{h}(\rho,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\ \delta(\rho - x\cos\theta - y\sin\theta)\ dx\ dy \tag{56}$$

$$= ab\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\tilde{x},\tilde{y})\ \delta(\rho - a\tilde{x}\cos\theta - b\tilde{y}\sin\theta)\ d\tilde{x}\ d\tilde{y} \tag{57}$$

$$= \frac{ab}{|\gamma|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\tilde{x},\tilde{y})\ \delta\left(\frac{\rho}{\gamma} - \tilde{x}\frac{a\cos\theta}{\gamma} - \tilde{y}\frac{b\sin\theta}{\gamma}\right)\ d\tilde{x}\ d\tilde{y} \tag{58}$$

$$\equiv \frac{ab}{|\gamma|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\tilde{x},\tilde{y})\ \delta(\tilde{\rho} - \tilde{x}\cos\tilde{\theta} - \tilde{y}\sin\tilde{\theta})\ d\tilde{x}\ d\tilde{y} \tag{59}$$

$$= \frac{ab}{|\gamma|}\ \breve{g}(\tilde{\rho},\tilde{\theta}) \tag{60}$$

Now $\tilde{\rho}$ and $\tilde{\theta}$ must be found as functions of $\rho$ and $\theta$. From the equations above it is found that

$$\cos\tilde{\theta} = \frac{a\cos\theta}{\gamma} \text{ and } \sin\tilde{\theta} = \frac{a\sin\theta}{\gamma} \tag{61}$$

$$\cos^2\tilde{\theta} + \sin^2\tilde{\theta} = 1 \Rightarrow \gamma = \sqrt{(a\cos\theta)^2 + (b\sin\theta)^2} \Rightarrow \tag{62}$$

$$\tilde{\rho} = \frac{\rho}{\sqrt{(a\cos\theta)^2 + (b\sin\theta)^2}} \tag{63}$$

$$a\tan\tilde{\theta} = b\tan\theta \Rightarrow \tilde{\theta} = \arctan\left(\frac{b}{a}\tan\theta\right) \tag{64}$$

To summerize the equations shown above

$$h(x,y) = g\left(\frac{x}{a}, \frac{y}{b}\right), \ a > 0 \text{ and } b > 0 \Rightarrow \tag{65}$$

$$\check{h}(\rho,\theta) = \frac{ab}{\gamma}\,\check{g}\left(\frac{\rho}{\sqrt{(a\cos\theta)^2 + (b\sin\theta)^2}}, \arctan\left(\frac{b}{a}\tan\theta\right)\right) \tag{66}$$

If using $0 \le \theta < \pi$, then the arctan function is

$$\arctan\left(\frac{b}{a}\tan\theta\right) = \begin{cases} \mathrm{Arctan}\left(\frac{b}{a}\tan\theta\right) & \text{if } 0 \le \theta < \frac{\pi}{2} \\ \mathrm{Arctan}\left(\frac{b}{a}\tan\theta\right) + \pi & \text{if } \frac{\pi}{2} \le \theta < \pi \end{cases} \tag{67}$$

The effect is illustrated on Figure 22 and 23.

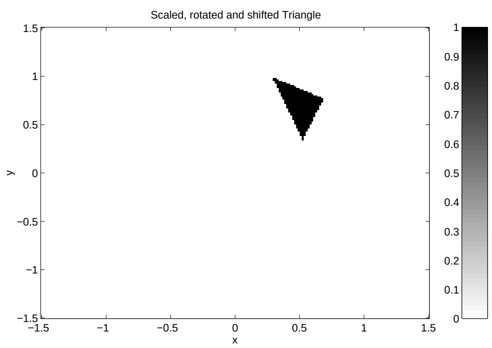


Figure 22: Scaling, rotating and shifting a triangle.

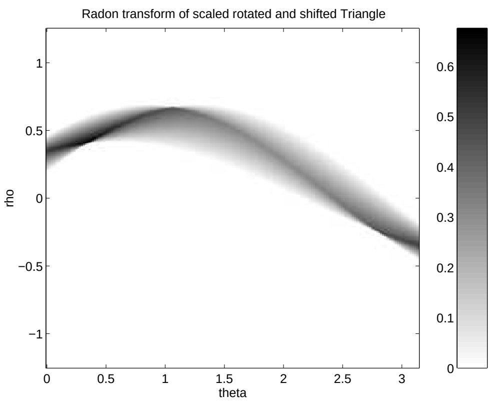


Figure 23: The corresponding Radon transform

# B An Example of an Initialization File

Here the `.INI`-file used to generate the man is shown.

```
OutFileName=man.fif
XSamples=401          Number of X samples
DeltaX=0.04           Sampling Distance in X
RhoSamples=351        Number of rho samples
OverSamp=0            Half oversampling factor.
GenerateImage=1       Generates image if 1. Optional -> Generates if omitted
GenerateRadon=1       Generates sinogram if 1. Optional -> Generates if omitted
DeltaRho=0.05         Sampling distance in rho
ThetaSamples=300      Number of theta samples. Distributed from 0 to pi.
NumberOfShapes=22     Number of simple elements.
DebugNiveau=_DDebug

        type  a      b     x0    y0      phi power  description

Shape0=  1   1      1.5    0.0   3.7      0   0.8   head
Shape1=  1   0.4    0.2    0.0   2.7      0  -0.2   mouth
Shape2=  2   1.5    2.0    0.0   0.0      0   1.0   stomach
Shape3=  2   1.5    0.3    0.0  -1.0      0  -0.3   belt
Shape4=  2   1.5    0.5    3.3   2.5     30   1.0   right arm
Shape5=  2   1.5    0.5   -2.4   0.6   -100   1.0   left arm
Shape6=  4   1      1.0    0.0   0.0     90  -0.2   triangle on stomach
Shape7=  1   0.2    0.3   -1.2   3.7      0   0.8   left ear
Shape8=  1   0.2    0.3    1.2   3.7      0   0.8   right ear
Shape9=  2   1.5    0.15   0.0  5.35      0   0.5   lower part of hat
Shape10= 2   0.7    0.6    0.0   6.1      0   0.5   upper part of hat
Shape11= 1   0.3    0.2   -0.5   4.0      0  -0.8   left eye
Shape12= 1   0.3    0.2    0.5   4.0      0  -0.8   right eye
Shape13= 1   0.2    0.4    0.0   3.5      0  -0.4   nose
Shape14= 2   2.0    0.5    2.0  -4.0    -60   1.0   right leg
Shape15= 1   0.5    0.4    5.5   3.7     30   0.5   right hand
Shape16= 1   1      0.5    4.0  -6.5     10   0.5   left foot
Shape17= 2   2.0    0.5   -2.0  -4.0     60   1.0   left leg
Shape18= 1   0.5    0.4   -2.8  -1.6   -100   0.5   left hand
Shape19= 1   1      0.5   -4.0  -6.5    -10   0.5   right foot
Shape20= 4   1      0.5    5.5   4.2    -90   0.5   icecream cone in right hand
Shape21= 1   0.5    0.5    5.5   4.9      0   0.5   only one ice cream cube!
```