Detection of Lines with Wiggles using the Radon Transform

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ABSTRACT

The discrete Radon transform is a useful tool in image processing for detection of lines (or in general curves) in digital images.
One of the key properties of the discrete Radon transform is that a line in an image is transformed into a peak in the parameter domain, where the position of the peak corresponds to the line parameters.
What often is needed, is to determine whether a Radon transform based curve detection algorithm will work in presence of noise. Lately analytical expressions for the probability of detecting a curve in presence of additive noise has been analyzed, and in this paper another kind of noise is analyzed theoretically, namely that the lines in the images might not be perfectly linear but include some random misalignment, here called wiggles.

1. INTRODUCTION

The Radon transform can be defined in various ways. In general image processing the lines are often parameterized using normal parameters [1], and in seismic signal processing the τ-p transform or slant stack version of the Radon transform use slope p and offset τ to parameterize the lines [2, 3]. Here the last definition will be used.

$$\breve{g}(p,\tau) = \int_{-\infty}^{\infty} g(x, px+\tau)\ dx \tag{1}$$

One of the key properties of the (discrete) Radon transform is that a line in an image is transformed into a peak in the parameter domain [4], where the position of the peak corresponds to the line parameters. In this way, the Radon transform converts a difficult global detection problem in the image domain into a more easily solved local peak detection problem in the parameter domain. A simple approach to recover the line parameters could then be to use a simple thresholding of the discrete parameter domain.

For a discrete image $g(m,n)$, a nearest neighbour approximation of Eq. 1 will be used

$$\breve{g}(\alpha,\beta) = \sum_{m=0}^{M-1} g\left(m, [\alpha m + \beta]\right) \tag{2}$$

where $[\cdot]$ denotes rounding to nearest neighbour.

Many different sources of noise can be of practical interest. Lately analytical expressions for the probability of detecting a curve in presence of additive noise has been analyzed [5]. In that paper, it was shown the the peak value in the parameter domain corresponding to the lines in the image must exceed a certain value relative to the variance of the additive noise, if the lines should be detected. In the following another kind of noise is considered, namely that the lines in the images might not be perfectly linear but include some random misalignment, here called wiggles. This type of noise source will give a reduction of the peak value, and analytical expressions will be derived to predict the actual reduction.

It could be mentioned that the popular Hough transform is related to the Radon transform, and two transforms will be equivalent under certain assumptions, see Chapter 3 in [4].

2. LINES WITH WIGGLES

It will be assumed that a line in the image can be modelled as

$$g(m,n) = \delta(n - [\alpha^* m + \beta^* + \lambda]) \qquad (3)$$

where $\delta(\cdot)$ is the Kronecker delta function, and λ is a Gaussian distributed noise term $\lambda \in \mathcal{N}(0, \sigma^2)$, which determines the change in position of the line in the n-direction, and it will be assumed that the noise terms λ are uncorrelated as a function of m.

What now is considered, is to find the average value (and shape) of the peak (if any is found) in the discrete parameter domain as a function of the noise deviation σ. Eq. 3 implies that the probability of the sample $g(m,n)$ being 1 is given by

$$\begin{aligned} & P\{g(m,n) = 1\} \\ &= P\left\{n = [\alpha^* m + \beta^* + \lambda]\right\} \\ &= P\left\{-\frac{1}{2} < n - \alpha^* m - \beta^* - \lambda < \frac{1}{2}\right\} \end{aligned} \qquad (4)$$

and assuming that the sample point (m,n) is given from the nearest neighbour mapping in the discrete Radon transform, $n = [\alpha m + \beta]$, then the rounding function is modelled as an additive (noise) term ω

$$\begin{aligned} & P\{g(m,n) = 1\} \\ &= P\left\{-\frac{1}{2} < [\alpha m + \beta] - \alpha^* m - \beta^* - \lambda < \frac{1}{2}\right\} \\ &= P\left\{-\frac{1}{2} < m(\alpha - \alpha^*) + \beta - \beta^* + \omega - \lambda < \frac{1}{2}\right\} \\ &= P\left\{-\frac{1}{2} < \zeta + \omega - \lambda < \frac{1}{2}\right\} \\ &= \Phi\left(\frac{\frac{1}{2} + \zeta + \omega}{\sigma}\right) - \Phi\left(\frac{-\frac{1}{2} + \zeta + \omega}{\sigma}\right) \end{aligned} \qquad (5)$$

$$\zeta = (\alpha - \alpha^*)m + \beta - \beta^* \qquad (6)$$

where $\Phi(\cdot)$ is the Gaussian probability function, and ζ is a displacement between the true line parameters (α^*, β^*) and the parameters (α, β) at a given position m.

Due to many values of m used in Eq. 2, it is a reasonable approximation to model ω as a uniformly distributed variable between the limits $-1/2$ and $1/2$, i.e., $\omega_m \in \mathcal{U}(-1/2, 1/2)$. Eq. 5 implies that the average contribution from the pixel $g(m,n)$ to the discrete Radon transform is given by

$$E\{\breve{g}(\alpha,\beta)|g(m,n)\} = \int_{\omega=-\frac{1}{2}}^{\omega=\frac{1}{2}} P_m(\omega)\; d\omega = \qquad (7)$$

$$\int_{\omega=-\frac{1}{2}}^{\omega=\frac{1}{2}} \Phi\left(\frac{\frac{1}{2} + \zeta + \omega}{\sigma}\right) - \Phi\left(\frac{-\frac{1}{2} + \zeta + \omega}{\sigma}\right) d\omega$$

In order to get an analytical expression for $E\{\breve{g}(\alpha,\beta)|g(m,n)\}$ the Gaussian probability function is approximated

$$\Phi(x) \approx \frac{1}{2}\left(1 + \tanh\left(\frac{x}{\gamma}\right)\right) \text{ where } \gamma = \sqrt{\frac{\pi}{2}} \qquad (8)$$

i.e., the integral of $\Phi(\cdot)$ can be approximated by

$$\int \Phi(x)\; dx \approx \frac{\gamma}{2}\left(\frac{1+x}{\gamma} + \log\cosh\left(\frac{x}{\gamma}\right)\right) \qquad (9)$$

where $\log(\cdot)$ is the natural logarithm.

If Eq. 9 is inserted into Eq. 7, followed by some rearrangements of the expressions, it is found that

$$E\{\breve{g}(\alpha,\beta)|g(m,n)\} = \\ \frac{\gamma\sigma}{2}\log\left(\frac{\cosh\left(\frac{2\zeta}{\gamma\sigma}\right) + \cosh\left(\frac{2}{\gamma\sigma}\right)}{\cosh\left(\frac{2\zeta}{\gamma\sigma}\right) + 1}\right) \qquad (10)$$

In Fig. 1 is the average weight to the discrete Radon transform shown as a function of σ and ζ.

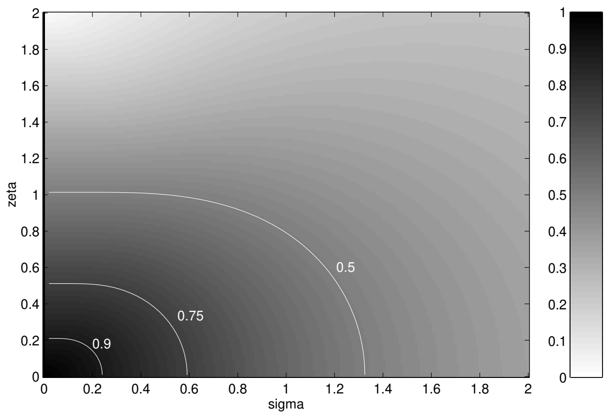

Fig. 1: *The average weight to the discrete Radon transform $E\{\breve{g}(\alpha,\beta)|g(m,n)\}$ as a function of the noise deviation σ on the first axis, and the displacement ζ on the second axis.*

A special case of interest regards $\zeta = 0$, i.e., where the coordinates of the line in the image matches exactly a sample in the discrete parameter domain.

$$\zeta = 0 \;\Rightarrow\; E\{\breve{g}(\alpha,\beta)|g(m,n)\} = \sigma\sqrt{\frac{\pi}{8}}\log\left(\frac{1}{2}\left(1+\cosh\left(\frac{\sqrt{8}}{\sigma\sqrt{\pi}}\right)\right)\right) \quad (11)$$

In Fig. 2 the average weight to the discrete Radon transform is shown as a function of σ, when the displacement ζ is assumed negligible. It can be seen that the function agrees well with a simulated result shown in Fig. 3, which is found by generating 10000 images for each noise amplitude σ and computing the discrete Radon transform only for the parameter set matching the true parameters, i.e., $\alpha = \alpha^*$ and $\beta = \beta^*$. An even better agreement can be found using a numerical integration of Eq. 7, but Eq. 9 provides an simple analytical result that approximates the simulated results well, though it can also be seen that the theoretical model has a small bias, when $\sigma \approx 0.1$.

Fig. 2 or Eq. 11 can be used to predict whether line parameters might be estimated given that the lines have wiggles. If, e.g., demanding that the peak value cannot decrease more than 50% due to wiggles, then σ cannot exceed approximately 0.7. And a 25% decrease of the peak value is found if $\sigma \approx 0.25$.

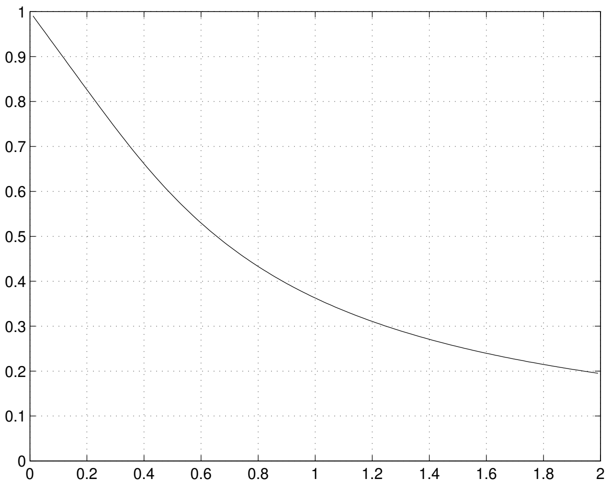

Fig. 2: *The theoretical result of the average weight to the discrete Radon transform, found in Eq. 11, as a function of σ when the displacement ζ is negligible.*

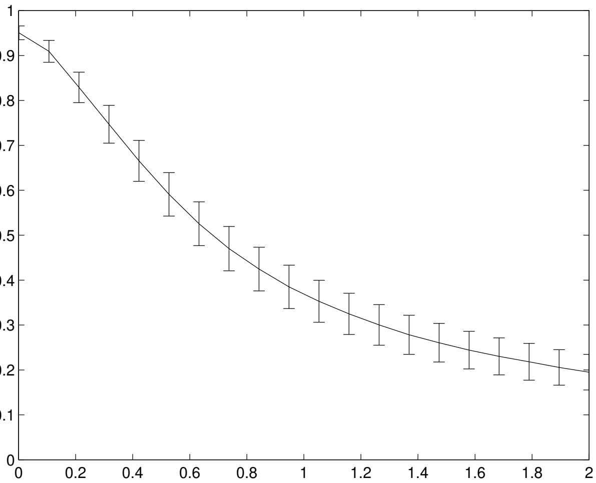

Fig. 3: *The simulated average value of the peak using 10000 images for each noise level. The vertical error bars show the measured deviation.*

3. EXAMPLES

In Fig. 4 is shown an image containing one line without noise, and Fig. 5 shows the discrete parameter domain in the area of the peak. Next, Fig. 6 shows an image where $\sigma = 0.5$, and Fig. 7 shows the corresponding discrete parameter domain. On the figure-axes, the continuous variables, indicated in Eq. 1, have been used. This merely imply a linear scaling of the discrete parameters used in Section 2. The images have been scaled individually according to the minimal and maximal values, and it can be seen that the maximal value here is around 65, and the shape is more uneven than seen from Fig. 5. This value lies within the error bars shown in Fig. 3.

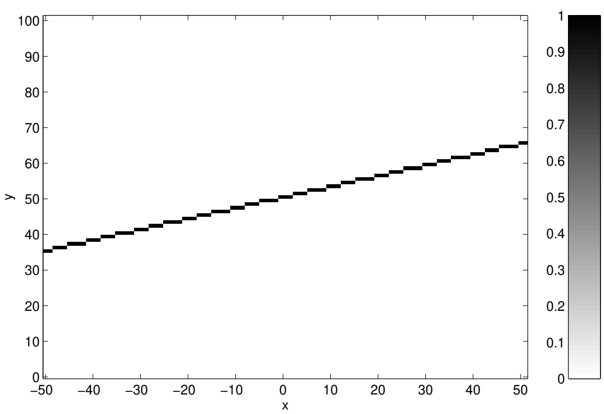

Fig. 4: *An image containing one line without any kind of noise.*

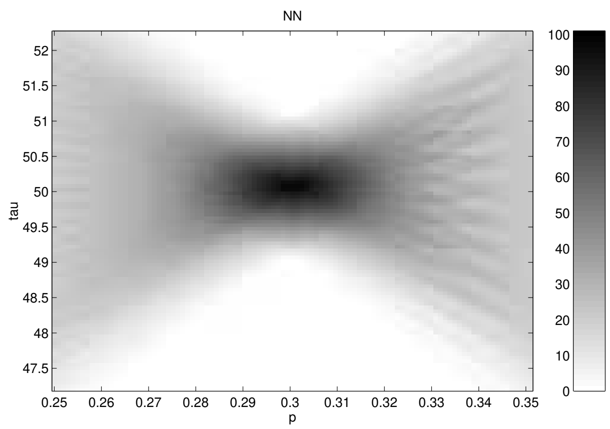

Fig. 5: *The discrete Radon transform of the image shown in Fig. 4 zoomed in at the peak.*

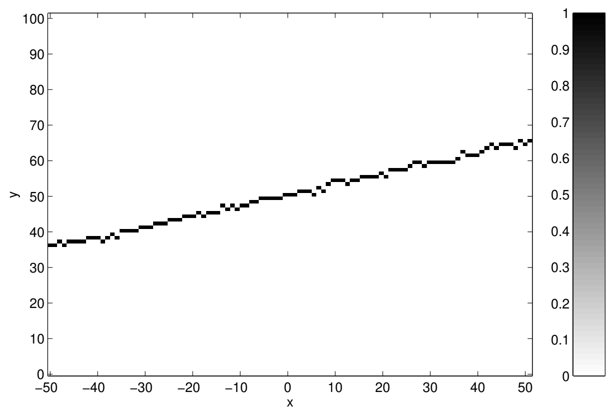

Fig. 6: *An image containing one wiggly line where* $\sigma = 0.5$.

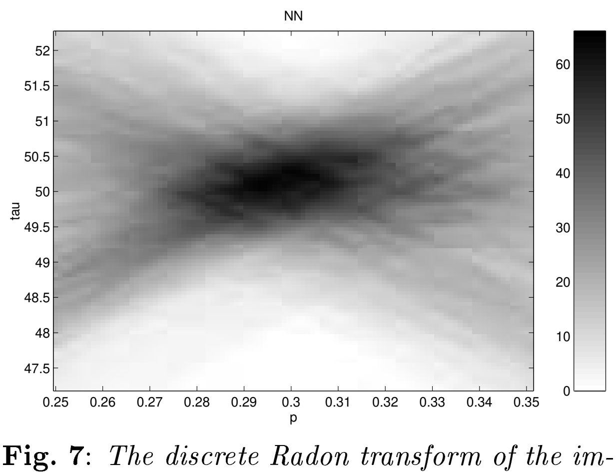

Fig. 7: *The discrete Radon transform of the image shown in Fig. 6 zoomed in at the peak.*

4. CONCLUSION

This paper has focused on the use of the Radon transform for line detection, assuming only one of many possibe noise sources, namely that the lines have wiggles. The conclusion is that a peak is generated in the discrete parameter domain corresponding to each of the wiggly line in the image, and the value of the peak will be lowered corresponding to the noise level. Analytical expressions have been given to determine the actual reduction of the peak value as a function of the noise level.

REFERENCES

[1] S. R. Deans. *The Radon Transform and Some of Its Applications.* Krieger Publishing Company, Malabar, Florida, 2 edition, 1993. Previously John Wiley & Sons Inc., 1983.

[2] Gregory Beylkin. Discrete Radon Transform. *IEEE Tr. ASSP*, 35(2):162–172, 1987.

[3] T. S. Durrani and D. Bisset. The Radon Transform and its Properties. *Geophysics*, 49(8):1180–1187, Aug. 1984. Correction note in vol. 50, no. 5, pp. 884–886, May 1985.

[4] Peter Toft. *The Radon Transform - Theory and Implementation.* PhD thesis, Department of Mathematical Modelling, Technical University of Denmark, 1996.

[5] Peter Toft. Using the Generalized Radon Transform for Detection of Curves in Noisy Images. In *Proceedings*, volume 4, pages 2221–2225. IEEE ICASSP, May 1996.

THE AUTHOR

Peter Toft was born in Virum, Denmark in 68. He received the M.Sc. in electrical engineering from the Technical University of Denmark, Lyngby, Denmark in 93.

Since 93, he has been employed at the Department of Mathematical Modelling (former Electronics Institute) at the Technical University of Denmark, where he recently finished his Ph.D. thesis. Currently he is employed in a post doc position, where functional brain imaging and reconstruction algorithms are in focus.

His research interests include curve detection, seismic signal processing, the Radon and Hough transforms, medical imaging, adaptive reconstruction methods, and 3D visualization.